Transition to deterministic chaos in a periodically driven quantum system and breaking of the time-reversal symmetry

U. Schwengelbeck and F. H. M. Faisal
Fakultät für Physik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany
(Received 16 February 1996; revised manuscript received 14 January 1997)

The phenomenon of transition from regular to chaotic dynamics in a periodically driven quantum system is demonstrated. The associated quantum Lyapunov numbers are determined analytically. A numerical experiment is made to test the nature of time evolution predicted by the theory. In contrast to the evolution in the regular domain, the passage to deterministic quantum chaos is found to break the time-reversal symmetry of the quantum dynamics, whenever the latter cannot be followed with infinite precision.
[S1063-651X(97)00705-8]

PACS number(s): 03.65.Sq, 05.45.+b, 05.30.Ch

Chaotic behavior in classical Hamiltonian systems, where local instabilities lead to an extremely sensitive dependence on the initial condition, has led to growing attention on the question of similar behavior in quantum Hamiltonian systems (e.g., [1]). In order to make possible a rigorous characterization of irregular (chaotic) dynamics in both the systems, a unified definition of Lyapunov characteristic numbers and Kolmogorov-Sinai (KS) entropy, based on the Hamilton-Jacobi formulation of the quantum mechanics, has been given recently [2]. An application of this definition to the quantum standard map has confirmed (in terms of vanishing Lyapunov exponents) the quantal suppression of chaos in this system [2] that was predicted earlier [3] and has been observed recently [4]. It has been shown [5] that a quantum version of Arnold's cat map [6], proposed by Weigert [7], provides a definite example of deterministic quantum chaos in terms of a positive Lyapunov exponent.

The aim of this work is to report on the phenomenon of transition from regular to chaotic quantum dynamics, accompanied by a breaking of the time-reversal symmetry of the wave function, whenever the dynamics cannot be followed with infinite precision. Before proceeding further, we shall briefly outline the definition of quantum Lyapunov exponents [2], using the Hamilton-Jacobi formulation of quantum mechanics, and then introduce the system of interest, namely, a generalized version of the quantum cat map of Weigert [7]. The wave function of a particle of charge q in an electromagnetic field is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi=\hat{H}\psi, \tag{1}$$

where the Hamiltonian is

$$\hat{H}=-\frac{\hbar^2}{2m}\left(\nabla-\frac{iq}{\hbar c}\mathbf{A}\right)^2+q\phi+V; \tag{2}$$

$\mathbf{A}$ and ϕ are the vector and scalar potential, respectively, and V is an external potential. Substitution of the wave function in the form $\psi=Re^{iS/\hbar}$ in Eq. (1) yields the generalized Hamilton-Jacobi equation

$$\partial S/\partial t+m\mathbf{v}^2/2+q\phi+V+Q=0 \tag{3}$$

and the continuity equation (with the density $P=R^2$)

$$\partial P/\partial t+\nabla\cdot(P\mathbf{v})=0, \tag{4}$$

where $\mathbf{v}=(\nabla S-q/c\mathbf{A})/m$ denotes the velocity field and $Q=-\hbar^2(\nabla^2R)/2mR$, appearing in Eq. (3), is the so-called quantum potential. The associated quantum trajectories of the particle are then governed by the equation of motion [8]

$$m\dot{\mathbf{x}}(t)=\nabla S(\mathbf{x},t)-(q/c)\mathbf{A}(\mathbf{x},t) \tag{5}$$

or, equivalently, by the quantum Newton equation [8]

$$m\ddot{\mathbf{x}}(t)=q\mathbf{E}+(q/c)\mathbf{v}\times\mathbf{B}-\nabla(V+Q). \tag{6}$$

It is noted that the present formulation of quantum mechanics, which goes back to de Broglie and was completed by Bohm [8], is fully consistent with the predictions of the conventional quantum mechanics. This is ensured by the fact that the density distribution $P_{\{\mathbf{x}(t)\}}(\mathbf{x},t)$ of the trajectories of the quantum ensemble, $\{\mathbf{x}(t)\}$, that evolves from an initial distribution of positions, given by $P_{\{\mathbf{x}(0)\}}(\mathbf{x},t_0)$, is equivalent to the Born probability density $|\psi(\mathbf{x},t)|^2$. A consequence of this formulation is the existence of a corresponding *non-negative* quantum phase space distribution function, given by

$$f(\mathbf{p},\mathbf{x},t)=P(\mathbf{x},t)\,\delta(\mathbf{p}-\nabla S(\mathbf{x},t)). \tag{7}$$

It is important to note that this formulation of the quantum theory provides also an unambiguous classical limit of the quantum dynamics. From Eq. (6) it can be seen that the condition for this limit is given by the vanishing of the quantum force: $-\nabla Q=\hbar^2\nabla[\nabla^2R/(2mR)]=0$, which is ensured if (i) $\hbar^2/m\to 0$ or (ii) $\nabla(R^{-1}\nabla^2R)\to\mathbf{0}$. Under these circumstances the quantum Newton equation (6) goes over directly to the corresponding classical one. This allows one to give a unified definition of the Lyapunov exponents λ in terms of the Euclidean distance $d(t)=\sqrt{\delta\mathbf{p}(t)^2+\delta\mathbf{x}(t)^2}$ in phase space for both classical and quantum dynamics [2]:

$$\lambda=\lim_{\substack{t\to\infty\\ d(t_0)\to 0}} t^{-1}\ln[d(t)/d(t_0)]. \tag{8}$$

This describes the asymptotic rate of exponential divergence of two neighboring trajectories, evolving from an initial distance $d(t_0)\rightarrow 0$ in the phase space. Thus, chaotic dynamics, associated with an extreme sensitivity to the initial condition, is characterized by $\lambda>0$, for both classical and quantum dynamics. To investigate the transition from regular to chaotic dynamics, we consider a quantum system consisting of a charged particle confined in a square of dimension $[0,L]\times[0,L]$, $L=1$, that is subjected to a periodic electromagnetic field, given by the vector potential

$$\mathbf{A}(\mathbf{x},t)=-(cm/q)\ \mathbf{V}\mathbf{x}\delta_\tau(t), \tag{9}$$

where $\delta_\tau(t)=\Sigma_{j=-\infty}^{\infty}\delta(t-j\tau)$, τ is the period length, and $\mathbf{x}=(x,y)^{\mathrm{T}}$ (mod1). We define matrix $\mathbf{V}$ in Eq. (9) via the transformation matrix

$$\mathbf{M}(K)=e^{\mathbf{V}}=\begin{pmatrix} 1 & K \\ 1 & K+1 \end{pmatrix}, \tag{10}$$

where $\det \mathbf{M}(K)=1$ ensures the corresponding map to be area preserving, and the system parmeter K is assumed to vary on the real axis. The scalar field ϕ may be chosen in analogy with [7] to reduce the Hamiltonian (2) in the Schrödinger equation into

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2-\frac{i\hbar}{2}(\nabla\cdot\mathbf{V}\ \mathbf{x}+\mathbf{x}\ \mathbf{V}\cdot\nabla)\,\delta_\tau(t). \tag{11}$$

It will be shown below that the parameter K, appearing in Eq. (10), controls the nature of the evolution of the wave function in time. In the special case $K=1$, our system reduces to the quantum realization of Arnold's cat map [6], due to Weigert [7], which has been shown [5] to exhibit deterministic quantum chaos in terms of a positive Lyapunov exponent λ. To be able to investigate the transition from regular to irregular quantum dynamics analytically, we consider below the so-called resonant case with a kick period of length $\tau=mL^2/\hbar\pi$. In this case the free evolution operator $U_0(\tau)=\exp(i\hbar\tau\nabla^2/2m)$ returns the wave function at the end of a period to its value at the beginning of the period. The time evolution of the periodically kicked system is then obtained by the repeated application of the evolution operator

$$\hat{U}(\mathbf{V})=\exp[-(\nabla\cdot\mathbf{V}\mathbf{x}+\mathbf{x}\ \mathbf{V}\cdot\nabla)/2] \tag{12}$$

on an initial state $\psi_0(\mathbf{x},t_0)$. An algebraic calculation similar to that in [7,5] shows that the wave function immediately before the $(n+1)$th kick, as well as immediately after the nth kick, is given by

$$\psi(\mathbf{x},t_{n+1}^-)=\psi(\mathbf{x},t_n^+)=\psi_0(\mathbf{M}^{-n}\ \mathbf{x},t_0^+), \tag{13}$$

where the transformation matrix

$$[\mathbf{M}(K)]^{-n}=\begin{pmatrix} K+1 & -K \\ -1 & 1 \end{pmatrix}^n. \tag{14}$$

The quantum equation of the motion of the system is obtained from Eqs. (5) and (9) to be

$$\mathbf{p}(t)=\dot{\mathbf{x}}(t)=m^{-1}\nabla S(\mathbf{x},t)+\mathbf{V}\mathbf{x}\delta_\tau(t). \tag{15}$$

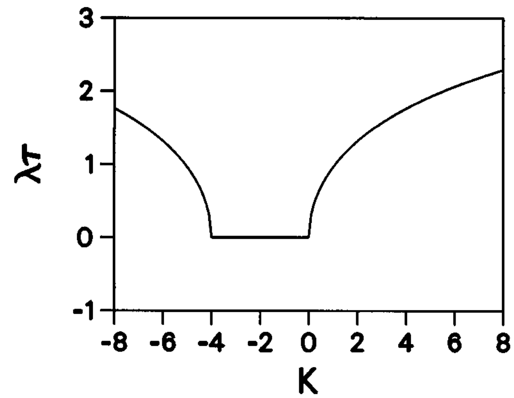

FIG. 1. Dimensionless Lyapunov number $\lambda\tau$ vs system parameter K, showing the domains of regular ($\lambda\tau=0$), $-4\leqslant K\leqslant 0$, and chaotic ($\lambda\tau>0$), $K<-4$ and $K>0$, dynamics.

It can be seen from Eq. (13) that for all real initial states $\psi_0(\mathbf{x},t_0^+)$, the wave function $\psi(\mathbf{x},t_n^{\pm})$ remains real and hence the gradient of the phase vanishes immediately before and after the nth kick for all n:

$$\mathbf{p}(t_n^{\pm})=\nabla S(\mathbf{x},t_n^{\pm})=\mathbf{0}. \tag{16}$$

Integration of the equation of motion (15) in the interval $t_n^-\leqslant t\leqslant t_n^+$ yields

$$\mathbf{x}(t)=e^{\mathbf{V}\ \theta(t-n\tau)}\mathbf{x}(t_n^-)\quad (\mathrm{mod}1). \tag{17}$$

Thus, the position of a single quantum trajectory at the time immediately after the nth kick, t_n^+, is given by

$$\mathbf{x}(t_n^+)=e^{\mathbf{V}}\ \mathbf{x}(t_n^-)=e^{\mathbf{V}}\ \mathbf{x}(t_{n-1}^+)\quad (\mathrm{mod}1), \tag{18}$$

where the last equality follows from the free propagation between two successive kicks under the resonance condition mentioned above. Given an initial coordinate $\mathbf{x}(t_0^+)$, repeated application of Eq. (18) yields [with $e^{\mathbf{V}}=\mathbf{M}(K)$]

$$\mathbf{x}(t_n^+)=[\mathbf{M}(K)]^n\ \mathbf{x}(t_0^+)\quad (\mathrm{mod}1). \tag{19}$$

From Eq. (19), the separation between two neighboring coordinates is given by

$$\delta\mathbf{x}(t_n^+)=[\mathbf{M}(K)]^n\ \ \delta\mathbf{x}(t_0^+); \tag{20}$$

the separation in canonical momenta in phase space, $\delta\mathbf{p}(t_n^+)$, vanishes at all times t_n^+ in view of Eqs. (7) and (16). Thus, the Euclidean phase space distance reduces to $d(t_n^+)=|\delta\mathbf{x}(t_n^+)|$. The Lyapunov exponent, defined by Eq. (8), then becomes

$$\lambda=\lim_{\substack{t_n^+\to\infty \\ |\delta\mathbf{x}(t_0^+)|\to 0}}\frac{1}{t_n^+}\ln\frac{|\delta\mathbf{x}(t_n^+)|}{|\delta\mathbf{x}(t_0^+)|}=\lim_{n\to\infty}\frac{1}{n\tau}\ln\|\mathbf{M}(K)\|^n, \tag{21}$$

where the last equality follows from Eq. (20) and $\|\ \|$ stands for the matrix norm. Thus, from the eigenvalues of the matrix $\mathbf{M}(K)$, given by

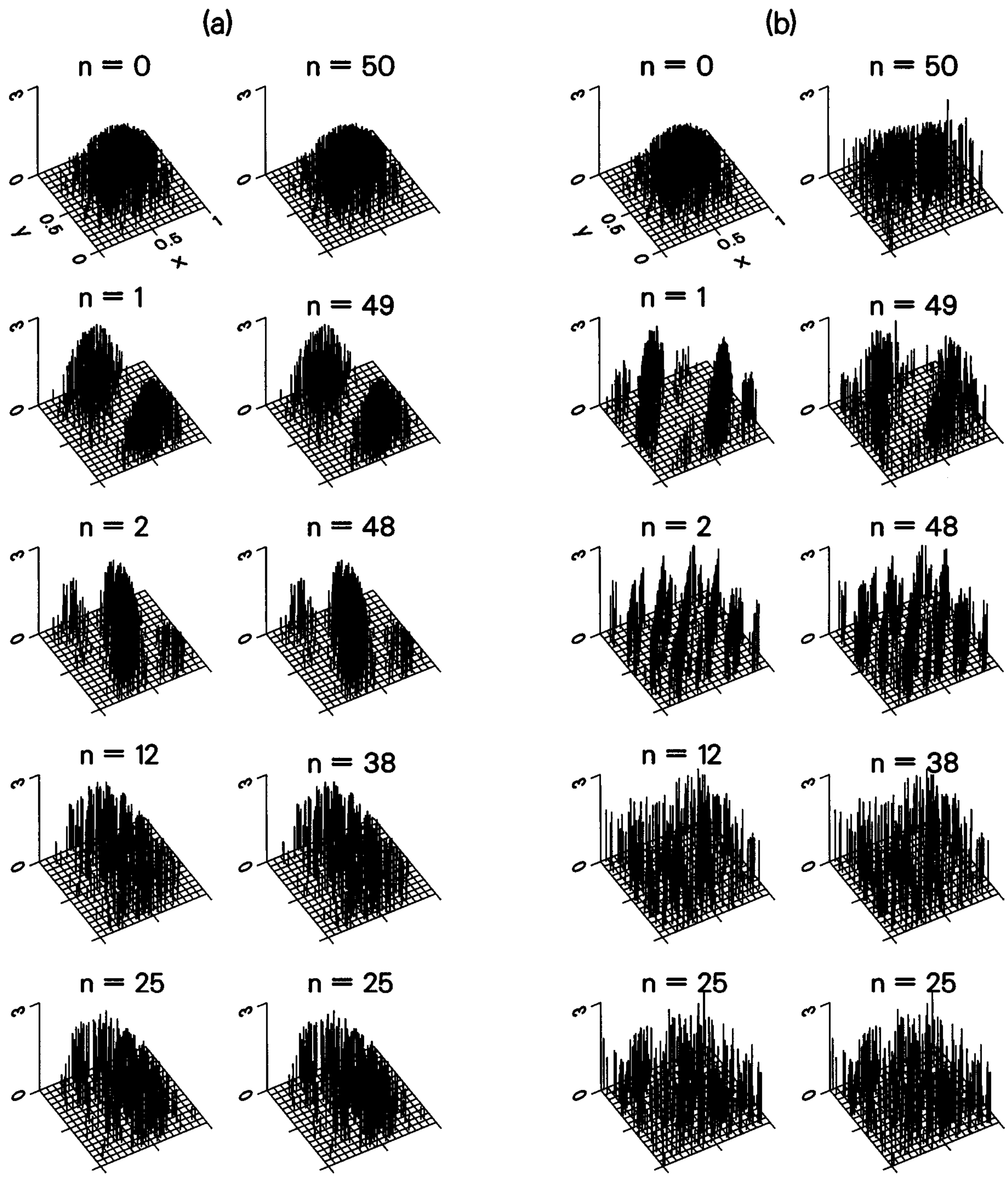

FIG. 2. Evolution of the wave function in a square of dimension [0,1]×[0,1], starting with the ground state and propagating up to the 25th period, then time reversing and propagating another 25 periods backward ($n=26,\ldots,50$). (a) Regular case with $K=0$, and (b) chaotic case with $K=2$. In case (a), the wave function maintains time reversibility and returns to its initial state, whereas in case (b), the time-reversal symmetry is broken (cf. text).

$$\gamma_{\pm}(K)=\frac{K+2}{2}\pm\sqrt{\left(\frac{K+2}{2}\right)^2-1}, \tag{22}$$

we finally obtain from (21) the quantum Lyapunov exponent

$$\lambda=\tau^{-1}\ln|\gamma(K)|, \tag{23}$$

where $|\gamma(K)|$ is the greater of $|\gamma_+(K)|$ and $|\gamma_-(K)|$. It can be seen from Eqs. (22) and (23) that for $-4\leqslant K\leqslant 0$, λ is zero, establishing that the dynamics of the system is regular in this domain. For parameter values $K<-4$ as well as $K>0$, the quantum Lyapunov exponent λ is positive definite, proving that the dynamics in these domains is rigorously chaotic. This regular to chaotic transition as a function of K is depicted in Fig. 1. It can be seen from this figure that the critical values of the parameters for this transition are at $K=0$ and $K=-4$. One of the practical consequences of the chaotic evolution is the long-time unpredictability of the dynamics. This fact can reflect itself in the breaking of the time-reversal symmetry of the quantum evolution in the chaotic region, whenever the latter cannot be followed with infinite precision. To test this prediction based on the analytical demonstration of deterministic chaos given above, we show below the results of numerical simulations of the propagation of the wave function of the system at a sequence of kick periods $n\tau$, starting with the unperturbed ground state

$$\psi_0(x,y,t_0^+)=2\sin(\pi x)\sin(\pi y). \tag{24}$$

Figures 2(a) and 2(b) show the evolution of the wave function during the first 25 kicks [the left-hand columns in (a) and (b), from above, downward], as well as the time-reversed evolution from the 25th period backward [the right-hand columns in (a) and (b), from below, upward]. Figure 2(a) corresponds to $K=0$, and therefore to the critical regular value of the quantum Lyapunov exponent $\lambda=0$. In this regular domain one expects a stable evolution in time for both the forward and backward propagations. This is indeed what is seen to be the case in this figure; not only does the forward evolution remain regular in time, but also the time-reversed evolution after the 25th kick brings back the wavefunction to its initial state. Figure 2(b) corresponds to a value of $K=2$ that lies in the chaotic domain with $\lambda>0$. It can be seen that not only does the wave function of the system become visibly chaotic with time [left-hand column of Fig. 2(b)] but also, on backward propagation from the 25th period, the system fails to return to its initial state [right-hand column of Fig. 2(b)], revealing a breakdown of the time-reversal symmetry in the chaotic region. The significance of this result can be appreciated by an examination of the connection between the Lyapunov number λ with the notion of algorithmic information, which are related, according to the Alekseev-Brudno theorem [9] as $\lambda=\lim_{t\to\infty}I(t)/t$, where $I(t)$ is the information necessary to record a stretch of the trajectory exactly in the interval of time t. In the long run, for a positive definite value of λ, the need for information increases boundlessly and the evolution cannot be followed exactly either in the forward or backward direction of motion, resulting in a breaking of time-reversal symmetry. Thus, the presence of quantum chaos provides an intrinsic mechanism for the origin of irreversibility in the realization of quantum dynamics. The length of time $t_{\rm crit}=n_{\rm crit}\tau$, over which the evolution can be recorded (not just imagined) is given by the Chirikov condition [1], $r=\lambda|t_{\rm crit}|/|\ln(\mu)|\leqslant 1$, where μ is the accuracy of recording. One may estimate therefore that for an accuracy $\mu\approx 10^{-14}$ and $\lambda\tau\simeq 1.32$, as in the case of Fig. 2(b), the critical time is about $n_{\rm crit}\approx 24$ periods, which is essentially the same as seen in Fig. 2(b).

To conclude, we have analyzed the regular to chaotic transition in a periodically driven two-dimensional quantum system, using a recently proposed quantum definition of Lyapunov exponent. The prediction of deterministic quantum chaos by the theory is tested in a numerical experiment: it is shown that the time-reversal symmetry of the wave function is broken by the onset of quantum chaos, whenever the evolution cannot be followed with infinite precision. It demonstrates that rigorous quantum chaos provides an intrinsic mechanism towards quantum irreversibility, independently of the presence of any influence from outside the system, such as that of ''baths,'' ''environmental dephasing,'' or ''measurements,'' and ''collapse'' of the wave function.

[1] *Quantum Chaos: Between Order and Disorder*, edited by G. Casati and B. V. Chirikov (Cambridge University Press, Cambridge, 1995).

[2] F.H.M. Faisal and U. Schwengelbeck, Forschungszentrum Bielefeld-Bochum-Stochastik, Universität Bielefeld, BiBoS No. 680 12/94, 1994 (unpublished); U. Schwengelbeck and F. H. M. Faisal, Phys. Lett. A **199**, 281 (1995).

[3] G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastical Behaviour in Classical and Quantum Hamiltonian Systems*, Lecture Notes in Physics Vol. 93 (Springer, Berlin, 1979).

[4] F. L. Moore, J. C. Robinson, C. F. Bharucha, Bala Sundaram, and M. G. Raizen, Phys. Rev. Lett. **75**, 4598 (1995).

[5] F. H. M. Faisal and U. Schwengelbeck, Phys. Lett. A **207**, 31 (1995).

[6] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, Reading, MA, 1968).

[7] S. Weigert, Z. Phys. B **80**, 3 (1990); Phys. Rev. A **48**, 1780 (1993).

[8] D. Bohm, Phys. Rev. **85**, 166 (1952); L. de Broglie, *Nonlinear Wave Mechanics: A Causal Interpretation* (Elsevier, Amsterdam, 1960).

[9] V. M. Alekseev and M. V. Yakobson, Phys. Rep. **75**, 287 (1981).